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On the Efficiency of Market Equilibrium in Production Economies

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Abstract. This paper introduces the notion of Market Equilibrium With Active Consumers (MEWAC), in order to characterize the efficiency of market outcomes in production economies. We show that, no matter the behaviour followed by the firms, a market equilibrium is efficient if it is a MEWAC. And also that every efficient allocation can be decentralized as a MEWAC in which firms follow the marginal pricing rule.

Key words: Efficiency, Imperfect competition, Market equilibrium, Production economies

1. Introduction

The standard general equilibrium model shows that perfectly competitive markets permit the spontaneous and efficient coordination of economic agents. In this context decentralization is neither chaotic nor wasteful. This is the *Invisible Hand Theorem*, according to Adam Smith's poetic expression. This Theorem relies on three key axioms about the nature of the economy: complete markets, price-taking behaviour and the convexity assumption.

The axiom of *complete markets* postulates that all agents face as many relative prices as necessary to solve their individual optimization problems. This amounts to saying that each commodity has associated with it a price and also that there are no *spill-overs* (externalities or public goods, say). In this case all non-price variables affecting individual agents' decision problems belong to their individual choice sets and efficiency only requires the equalization of *private* marginal rates to relative prices.

Price taking behaviour means that those variables conditioning agents' choices are independent on their individual actions. Therefore, individual maximization entails the equalization of marginal rates with the corresponding prices. When markets are complete, this axiom implies that in equilibrium all commodities will have the same marginal value in all possible uses and for all agents.

The convexity assumption says that agents choose actions within convex sets, guided by a convex criterion (a quasi-concave objective function). From this it follows that marginal conditions are sufficient to characterize the behaviour of individual agents, because all local maxima turn out to be global, and the necessary conditions for a maximum are also sufficient. Moreover, agents' behaviour can be described in terms of upper hemicontinuous correspondences, with non-empty, closed and convex values. This, together with the other axioms, permits one to apply a fixed-point argument in order to prove the existence of equilibrium.

Therefore, complete markets, parametric prices and convexity imply that the set of competitive equilibria of a given economy is non-empty (existence of equilibrium) and coincides with the set of efficient allocations (the two welfare theorems). The equalization of private marginal rates is a sufficient condition to ensure the efficiency of equilibrium allocations, because the local properties that characterize the maximization of individual objective functions imply global maximization of the aggregate payoffs. These three axioms also provide us with precise guidelines about the environments in which we can expect *market failures*: Incomplete markets (including here the cases of externalities and public goods), monopolistic competition, and increasing returns to scale or other forms of non-convexities.

The purpose of this paper is to analyze the relationship between equilibrium and efficiency in an imperfectly competitive scenario. We aim at providing some insights on the presence of market failures in economies in which production sets are not assumed to be convex and/or firms do not behave competitively, while keeping the assumption of complete markets. The analysis relies on two key methodological features:

(i) Modelling the behaviour of firms by means of abstract *pricing rules.*

(ii) Extending the role played by consumers in production decisions.

A pricing rule is a mapping from the firm's set of efficient production plans to the price space. The graph of such a mapping describes the prices-production pairs which a firm finds *acceptable*. An equilibrium for the economy is defined in this setting as a price vector, a list of consumption allocations, and a list of production plans such that: (a) consumers maximize their preferences subject to their budget constraints; (b) each individual firm is in "equilibrium" at those prices and production plans (namely, the prices-production combination is in the graph of the firm's pricing rule); and (c) the markets for all goods clear. It is the nature of the equilibrium condition (b) which establishes the difference with respect to the Walrasian model.

There are very general existence results for equilibrium models in which firms' behaviour is described in terms of abstract pricing rules (see, for instance, Bonnisseau and Cornet, 1988; Villar, 2000a, ch. 5). Moreover, Guesnerie (1975) has shown that every Pareto optimal allocation can be obtained as a marginal pricing equilibrium, regardless of the convexity of production sets. Under very mild regularity conditions marginal pricing is actually a necessary condition for optimality. Yet, marginal pricing does not ensure efficient outcomes. It may well be that there is an inadequate number of active firms in equilibrium, so that the resulting production lies in the interior of the aggregate production set (Beato and Mas-Colell, 1985). Even if we take the simplified case of a single firm, there are economies in which no marginal pricing equilibrium is Pareto optimal (Guesnerie, 1975; Brown and Heal, 1979), and economies in which marginal pricing is Pareto dominated by average cost pricing (Vohra, 1988). This is so because, contrary to the standard convex world, the mapping associating efficient allocations to income distributions is not *onto* (see the discussion in Guesnerie, 1990; Vohra, 1990, 1991; Quinzii, 1992, ch. 4). These conclusions indicate the presence of an *impossibility result:* marginal pricing is a necessary condition for optimality, but it does not yield efficient outcomes.

It is interesting to observe the role played by the consumers in this impossibility result. A characteristic feature of those equilibrium models with non-convex production sets is that consumers' income is defined as a mapping whose domain is the Cartesian product of the price space and the space of production allocations. That is to say, prices and production plans are treated as two independent sets of variables, regarding the consumer's choice problem. The strategy of including more variables to define the restrictions faced by the consumers helps demonstrating the existence of equilibrium. Yet, this procedure generates many equilibria in which efficiency cannot be achieved because reallocating the resources devoted to production activities might yield higher incomes at given prices (which clearly implies the inefficiency of the original allocation).

We propose the notion of Market Equilibrium With Active Consumers (MEWAC) as a way of retrieving the link between consumers' wealth and production decisions. A MEWAC is a situation in which all consumers agree on the production plans that firms are to develop, and all firms agree on the prices that are to be associated with these production plans. The "price agreement" is the standard requirement for an equilibrium, in those models where firms' behaviour is modelled in terms of pricing rules. The "production agreement" among consumers is new. It indicates that we abandon here the assumption that consumers adjust passively to the decisions made by the firms. They participate in production decisions trying to maximize their net income. This action translates the idea that the owners of the firms have a say on their decisions.

We show that, no matter the behaviour followed by the firms, a market equilibrium is efficient if it is a MEWAC. And also that every efficient allocation can be decentralized as a MEWAC in which firms follow the marginal pricing rule. This equilibrium notion is related to the concept of *valuation equilibria*, as used in Hammond and Villar (1998, 1999) for the analysis of economies with spillovers (incomplete markets).

The paper is organized as follows: Section 2 contains the model, section 3 the results, and a few final comments are gathered in Section 4.

2. The Model

2.1. ECONOMIES

Consider an economy with ℓ commodities, m consumers and n firms. The vector

$\omega \in \mathbb{R}^\ell$ represents the aggregate initial endowments. There are n firms in the economy. We denote by $Y_j \subset \mathbb{R}^\ell$ the jth firm's production set, and by $\mathbb{F}_j$ the jth firm's set of weakly efficient production plans, that is,

$$\mathbb{F}_j \equiv \{\mathbf{y}_j \in Y_j \mid \mathbf{y}'_j >> \mathbf{y}_j \Rightarrow \mathbf{y}'_j \notin Y_j\}$$

Let $\mathbb{F} \equiv \prod_{j=1}^n \mathbb{F}_j$ stand for the Cartesian product of the n sets of weakly efficient production plans. Points in $\mathbb{F}$ are denoted by

$$\widetilde{\mathbf{y}} = (\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n)$$

A *pricing rule* is an abstract construct that provides a general way of describing the behaviour of firms. A pricing rule for the jth firm can be defined as a mapping ϕ_j applying the set of efficient production plans $\mathbb{F}_j$ into $\mathbb{R}^\ell_+$. For a point $\mathbf{y}_j$ in $\mathbb{F}_j$, $\phi_j(\mathbf{y}_j)$ has to be interpreted as the set of price vectors found acceptable by the jth firm when producing $\mathbf{y}_j$. In other words, the jth firm is in equilibrium at $(\mathbf{p}, \mathbf{y}_j)$, if $\mathbf{p} \in \phi_j(\mathbf{y}_j)$. Formally:

DEFINITION 1. A *Pricing Rule* for the jth firm is a correspondence, $\phi_j : \mathbb{F}_j \rightarrow \mathbb{R}^\ell_+$.

Some familiar examples of pricing rules that exhibit good operational properties under standard assumptions are the following:

(i) *Profit maximization*, ϕ_j^{PM}. Assuming that production sets are convex, this pricing rule associates with each efficient production plan the corresponding set of supporting prices. Namely, $\phi_j^{PM}(\mathbf{y}_j) = \{\mathbf{p} \in \mathbb{R}^\ell_+ \ / \ \mathbf{p}\mathbf{y}_j \geqslant \mathbf{p}\mathbf{y}'_j, \ \ \forall \ \mathbf{y}'_j \in Y_j\}$.

(ii) *Average cost pricing*, ϕ_j^{AC}. This is a pricing rule that associates with each efficient production plan those prices that make the firm to break even. Formally (taking $\mathbf{y}_j \neq \mathbf{0}$), $\phi_j^{AC}(\mathbf{y}_j) = \{\mathbf{p} \in \mathbb{R}^\ell_+ \ / \ \mathbf{p}\mathbf{y}_j = 0\}$.

(iii) *Marginal pricing*, ϕ_j^{MP}. This pricing rule describes a situation in which firms sell their output at prices that satisfy the necessary conditions for optimality. $\phi_j^{MP}(\mathbf{y}_j)$ is usually associated with Clarke Normal Cone to Y_j at the boundary point $\mathbf{y}_j$.

(iv) *Two-part marginal pricing*. This is a non-linear price structure which combines marginal and average cost pricing, by charging an entrance fee plus a proportional one, to those consumers who buy positive amounts of the goods produced by non-convex firms.

(v) *Constrained profit maximization.* This is actually a family of pricing rules that describe a situation in which firms maximize profits at given prices, subject to some quantity constraints.

REMARK 1. One can define more generally a pricing rule as a correspondence, $\Phi_j : \mathbb{F} \times \mathbb{R}^\ell_+ \rightarrow \mathbb{R}^\ell_+$. In that case, the jth firm's pricing rule depends on the "market conditions", as summarized by all firms' production plans and a reference price vector.

There are m consumers in the economy, each characterized by a triple (X_i, u_i, r_i) where $X_i \subset \mathbb{R}^\ell$, $u_i : X_i \to \mathbb{R}$, denote the ith consumer's consumption set and utility function, and $r_i : \mathbb{R}^\ell_+ \times \mathbb{F} \to \mathbb{R}$ is the ith consumer's net income mapping, as a function of market prices and the firms' production plans. The net income mapping may include private income as well as taxes and transfers.

For the sake of illustration, we can think of the income mapping as given by:

$$r_i(\mathbf{p}, \widetilde{\mathbf{y}}) = \mathbf{p}\omega_i + \sum_{j=1}^{n} \theta_{ij} \mathbf{p}\mathbf{y}_j + \tau_i(\mathbf{p}, \widetilde{\mathbf{y}}) \tag{1}$$

where $\omega_i \in \mathbb{R}^\ell$ is the ith consumer's vector of initial endowments, θ_{ij} her share in the jth firm's profits (or losses, if negative), and $\tau_i : \mathbb{R}^\ell_+ \times \mathbb{F} \to \mathbb{R}$ describes the ith consumer's tax-subsidy mapping. It is assumed, by the very definition of a market economy, that $\sum_{i=1}^m \omega_i = \omega$ (that is, total resources are fully distributed among the consumers) and $\sum_{i=1}^m \theta_{ij} = 1$, for all j (that is, firms are owned by the consumers). This income mapping corresponds to a private ownership market economy with taxes and transfers. We shall refer here and there to this particular case.

Consider now the following definition:

DEFINITION 2. An *income schedule* is a collection of mappings $(r_i)_{i=1}^m$, with $r_i : \mathbb{R}^\ell_+ \times \mathbb{F} \to \mathbb{R}$ for all i, such that, for all $(\mathbf{p}, \widetilde{\mathbf{y}}) \in \mathbb{R}^\ell_+ \times \mathbb{F}$:
(i) $r_i(\lambda\mathbf{p}, \widetilde{\mathbf{y}}) = \lambda r_i(\mathbf{p}, \widetilde{\mathbf{y}})$, for all $\lambda > 0$.
(ii) $\sum_{i=1}^m r_i(\mathbf{p}, \widetilde{\mathbf{y}}) \leqslant \mathbf{p}\omega + \sum_{j=1}^n \mathbf{p}\mathbf{y}_j$.

DEFINITION 3. An income schedule is *balanced* when it satisfies $\sum_{i=1}^m r_i(\mathbf{p}, \widetilde{\mathbf{y}}) = \mathbf{p}\omega + \sum_{j=1}^n \mathbf{p}\mathbf{y}_j$ for all $(\mathbf{p}, \widetilde{\mathbf{y}}) \in \mathbb{R}^\ell_+ \times \mathbb{F}$.

An income schedule is a collection of mappings, one for each consumer, that satisfy two simple and intuitive properties. First, that each mapping is homogeneous of degree one in prices. Second, that total income cannot exceed the worth of the aggregate resources plus total profits. When part (ii) holds with equality the income schedule is said to be balanced. In the particular case illustrated by Equation (1) above, an income schedule requires all tax-subsidy mappings being homogeneous of degree one in prices and *self-financing*, that is, $\sum_{i=1}^m \tau_i(\mathbf{p}, \widetilde{\mathbf{y}}) \leqslant 0$ (resp. $\sum_{i=1}^m \tau_i(\mathbf{p}, \widetilde{\mathbf{y}}) = 0$ if balanced).

REMARK 2. Taking r_i as a mapping defined on the Cartesian product $\mathbb{R}^\ell_+ \times \mathbb{F}$ amounts to considering prices and production plans as two separate sets of variables, from the ith consumer's viewpoint (contrary to the case of standard competitive economies).

An **economy** is a collection of: (a) m consumers, each characterized by her consumption set, her utility function and her income mapping; (b) n firms, each characterized by its production set and its pricing rule; and (c) A vector ω of initial endowments. This can be written shortly as:

$$E = [(X_i, u_i, r_i)_{i=1}^m, (Y_j, \phi_j)_{j=1}^n, \omega].$$

Note that this definition permits different firms to follow different patterns of behaviour (embodied in their corresponding pricing rules, on which nothing is being assumed).

The following definition makes it precise the standard equilibrium notion for a market economy:

DEFINITION 4. A *market equilibrium* for an economy E, is a price vector $\mathbf{p}^* \in \mathbb{R}_+^\ell$, and an allocation $[(\mathbf{x}_i^*), \widetilde{\mathbf{y}}^*)]$ such that:

(i) For every $i = 1, 2, ..., m$, $\mathbf{x}_i^*$ maximizes u_i over the set of points $\mathbf{x}_i \in X_i$ such that $\mathbf{p}^*\mathbf{x}_i \leqslant r_i(\mathbf{p}^*, \widetilde{\mathbf{y}}^*)$.

(ii) $\mathbf{p}^* \in \bigcap_{j=1}^n \phi_j(\mathbf{y}_j^*)$.

(iii) $\sum_{i=1}^m \mathbf{x}_i^* - \omega = \sum_{j=1}^n \mathbf{y}_j^*$.

A *market equilibrium* is a price vector and a feasible allocation such that: (a) all consumers maximize utility at given prices within the budget sets that result from a passive adjustment to the firms' decisions; and (b) all firms are in equilibrium according to their pricing rules (which can differ from one another).

Market equilibria can be shown to exist when the following conditions hold (e.g. Bonnisseau and Cornet, 1988, th. 2′): (1) X_i is a non-empty, closed, and convex subset of $\mathbb{R}^\ell$, bounded from below; (2) u_i is continuous, quasi-concave and locally non-satiable; (3) r_i is continuous and satisfies the cheaper point requirement on the set of production equilibria;⋆ (4) Y_j is closed and comprehensive; (5) ϕ_j is upper hemicontinuous, with non-empty, closed and convex values, and bounded losses;⋆⋆ and (6) the set of attainable allocations is compact. The proof of this general existence result relies very much on the treatment of income mappings as functions that are defined on the Cartesian product $\mathbb{R}_+^\ell \times \mathbb{F}$, that is, functions that treat prices and production plans as two separate sets of variables (see Remark 2 above).

We denote by $\mathcal{A}(\omega)$ the set of attainable allocations, that is,

$$\mathcal{A}(\omega) = \left\{ [(\mathbf{x}_i)_{i=1}^m, \widetilde{\mathbf{y}}] \in \Pi_{i=1}^m X_i \times \Pi_{j=1}^n Y_j \;\Big/\; \sum_{i=1}^m \mathbf{x}_i - \sum_{j=1}^n \mathbf{y}_j \leqslant \omega \right\}$$

Let $\mathbb{F}^{\mathcal{A}}$ stand for the set of efficient and attainable production plans. That is, $\mathbb{F}^{\mathcal{A}}$ is the projection of $\mathcal{A}(\omega)$ over $\mathbb{F}$.

⋆ That is to say, $r_i(\mathbf{p}, \widetilde{\mathbf{y}}) > \min \mathbf{p}X_i$ for all $(\mathbf{p}, \widetilde{\mathbf{y}}) \in \mathbb{R}_+^\ell \times \mathbb{F}$ such that $\mathbf{p} \in \bigcap_{j=1}^n \phi_j(\mathbf{y}_j)$.

⋆⋆ That means that $\mathbf{q}\mathbf{y}_j \geqslant \alpha$, for all $\mathbf{q} \in \phi_j(\mathbf{y}_j)$, all $\mathbf{y}_j \in \mathbb{F}_j$, some given scalar $\alpha \leqslant 0$.

2.2. ACTIVE CONSUMERS

Let us now consider the case in which households are involved in the firms' production decisions. We refer to these households as *active consumers*. Given a price vector $\mathbf{p} \in \mathbb{R}_+^\ell$ the demand of an active consumer is obtained as a solution to the following program:

$$\left.\begin{array}{c} \max_{(\mathbf{x}_i,\widetilde{\mathbf{y}})} \;\; u_i(\mathbf{x}_i) \\ s.t.: \mathbf{x}_i \in X_i \\ \widetilde{\mathbf{y}} \in \mathbb{F}^{\mathcal{A}} \\ \mathbf{p}\mathbf{x}_i \leqslant r_i(\mathbf{p},\widetilde{\mathbf{y}}) \end{array}\right\} \tag{2}$$

Note that the variables on which the ith consumer is making a choice include her private consumption plan and the production plan of the firms. We denote by $\mathcal{Y}^i(\mathbf{p})$ the set of points $\widetilde{\mathbf{y}}^i \in \mathbb{F}^{\mathcal{A}}$ such that $(\mathbf{x}_i, \widetilde{\mathbf{y}}^i)$ solves program [2], for some $\mathbf{x}_i \in X_i$. Needless to say that $\mathcal{Y}^i(\mathbf{p})$ may differ from one consumer to another.

Now define the ith consumer's maximal income mapping as a function R_i : $\mathbb{R}_+^\ell \times \mathbb{F}^{\mathcal{A}} \rightarrow \mathbb{R}$ given by $R_i(\mathbf{p}) := r_i(\mathbf{p},\widetilde{\mathbf{y}}^i)$. That is, $R_i(\mathbf{p})$ is the income that the ith consumer "demands" at prices $\mathbf{p}$, by choosing those attainable production plans that maximize the net revenue of her assets at these prices. This notional income mapping re-establishes the relationship between prices and production in the consumers' choice problem. Indeed, this is the analog of competitive budget sets, that is, $R_i(\mathbf{p})$ is precisely the income that consumers obtain when convex firms maximize profits at given prices. Clearly, $R_i(\mathbf{p}) = r_i(\mathbf{p},\widetilde{\mathbf{y}})$ whenever $\widetilde{\mathbf{y}} \in \mathcal{Y}^i(\mathbf{p})$. Also observe that, for an arbitrary $\widetilde{\mathbf{y}} \in \mathbb{F}$,

$$\sum_{i=1}^m R_i(\mathbf{p}) = \sum_{i=1}^m r_i(\mathbf{p},\widetilde{\mathbf{y}}^i) \geqslant \sum_{i=1}^m r_i(\mathbf{p},\widetilde{\mathbf{y}})$$

with the equality holding if and only if $\widetilde{\mathbf{y}} \in \bigcap_{i=1}^m \mathcal{Y}^i(\mathbf{p})$. Note, however, that $\sum_{i=1}^m R_i(\mathbf{p}) \geqslant \sum_{j=1}^n \mathbf{p}\mathbf{y}_j + \mathbf{p}\omega$, for any given $\widetilde{\mathbf{y}} \in \mathbb{F}$, can only be ensured when the income schedule is balanced.

We are now ready to present the equilibrium concept for this economy:

DEFINITION 5. A *market equilibrium with active consumers* (MEWAC, for short) for an economy E is a price vector $\mathbf{p}^* \in \mathbb{R}_+^\ell$ and an allocation $[(\mathbf{x}_i^*), \widetilde{\mathbf{y}}^*]$ such that:

(i) For every $i = 1, 2, \ldots, m$, $\mathbf{x}_i^*$ maximizes u_i over the set of points $\mathbf{x}_i \in X_i$ such that $\mathbf{p}^*\mathbf{x}_i \leqslant r_i(\mathbf{p}^*, \widetilde{\mathbf{y}}^*)$.

(ii) $\widetilde{\mathbf{y}}^* \in \bigcap_{i=1}^m \mathcal{Y}^i(\mathbf{p}^*)$.

(iii) $\mathbf{p}^* \in \bigcap_{j=1}^n \phi_j(\mathbf{y}_j^*)$.

(iv) $\sum_{i=1}^m \mathbf{x}_i^* - \omega = \sum_{j=1}^n \mathbf{y}_j^*$.

(v) $\sum_{i=1}^m r_i(\mathbf{p}^*, \widetilde{\mathbf{y}}^*) = \mathbf{p}^*\omega + \sum_{j=1}^n \mathbf{p}^*\mathbf{y}_j^*$.

A *market equilibrium with active consumers* is a price vector and a feasible allocation such that: (a) no consumer finds it individually beneficial to choose an alternative consumption plan that is affordable, with respect to the maximum income achievable at the equilibrium prices; (b) firms follow their corresponding pricing rules and execute the consumers' production decisions; and (c) total income equals the worth of total resources plus total profits. Note that, when consumers are not satiable, (v) follows from (i), (ii) and (iv).

By definition, in a MEWAC $r_i(\mathbf{p}^*, \widetilde{\mathbf{y}}^*) = R_i(\mathbf{p}^*)$ for all i. A consumption plan $\mathbf{x}_i$ that maximizes utility subject to the restriction $\mathbf{p}\mathbf{x}_i \leqslant R_i(\mathbf{p})$ can be identified with the ith consumer's *notional demand*, whereas those $\mathbf{x}_i$ that maximize utility subject to $\mathbf{p}\mathbf{x}_i \leqslant r_i(\mathbf{p},\widetilde{\mathbf{y}})$ can be identified with her *effective demand.* From this point of view a MEWAC is a market equilibrium in which effective and notional demands coincide.

We can think of a MEWAC as the equilibrium of a market mechanism that works as follows. There is an auctioneer who calls out a price vector $\mathbf{p} \in \mathbb{R}^\ell_+$. For all $i = 1, 2, \ldots, m$, the ith consumer calculates the income she can achieve at these prices, by choosing the most convenient production plans for the firms she owns (taking into account the tax-subsidy rule included in her income mapping).⋆ This determines the set $\mathcal{Y}^i(\mathbf{p})$. Then she solves her demand problem by maximizing utility at these prices within the budget set associated with her wealth estimated in that way. For all $j = 1, 2, \ldots, n$, the jth firm aggregates the orders given by all the incumbent consumers in a target production plan (e.g. the point in $\mathbb{F}_j$ which is closest to the weighted average of the consumers' proposals, where the weights correspond to their respective property shares). Then, it chooses those prices that make that production plan agree with its objectives (as defined through its pricing rule). The auctioneer compares all these actions and modifies the reference price vector when these decisions are inconsistent, until an equilibrium is reached. An equilibrium is a fixed point of this process.

A market equilibrium with active consumers involves two unanimous agreements among the agents. On the one hand, all the firms agree on the price vector associated to the equilibrium production allocation, that is, $\mathbf{p}^* \in \bigcap_{j=1}^n \phi_j(\mathbf{y}_j^*)$. On the other hand, all the consumers agree on the production allocation that corresponds to the equilibrium prices, that is, $\widetilde{\mathbf{y}}^* \in \bigcap_{i=1}^m \mathcal{Y}^i(\mathbf{p}^*)$. From this it follows that a MEWAC imposes restrictions on the admissible pricing rules, restrictions that are related to the income schedule. Note however that this double agreement may be easier to achieve if one introduces the reasonable assumption that $\phi_j(\mathbf{y}_j) = \mathbb{R}^\ell_+$ whenever $\mathbf{y}_j$ is unanimously chosen by all incumbent consumers. This amounts to saying that the firms accept the decisions of their owners under the unanimity rule.

⋆ We assume implicitly here that the ith consumer's tax subsidy rule is independent on the production plans of those firms in which she has no participation.

3. The results

3.1. THE TWO WELFARE THEOREMS

We now proceed to analyze how this equilibrium concept fares with respect to the two welfare theorems.

Our first result establishes that a MEWAC yields an efficient allocation. Formally:

THEOREM 1. *Let E be an economy with locally non-satiable consumers. A MEWAC for E yields an efficient allocation, provided the income schedule is balanced.*

Proof. Let $[\mathbf{p}^*, (\mathbf{x}_i^*), \widetilde{\mathbf{y}}^*]$ be a MEWAC and suppose that $[(\mathbf{x}_i), \widetilde{\mathbf{y}}]$ is a feasible allocation such that $u_i(\mathbf{x}_i) \geqslant u_i(\mathbf{x}_i^*)$ for all i, with a strict inequality for some agent. As this allocation is feasible, it must be the case that $\sum_{i=1}^m \mathbf{x}_i \leqslant \omega + \sum_{j=1}^n \mathbf{y}_j$, and consequently,

$$\sum_{i=1}^{m} \mathbf{p}^*\mathbf{x}_i \leqslant \mathbf{p}^*\omega + \sum_{j=1}^{n} \mathbf{p}^*\mathbf{y}_j \tag{3}$$

It follows from the non-satiation hypothesis, the definition of MEWAC and the balancedness condition, that:

$$\sum_{i=1}^{m} \mathbf{p}^*\mathbf{x}_i > \sum_{i=1}^{m} R_i(\mathbf{p}^*) \geqslant \mathbf{p}^*\omega + \sum_{j=1}^{n} \mathbf{p}^*\mathbf{y}_j$$

But this contradicts expression (3) above. Therefore, such an allocation cannot exist. □

Theorem 1 establishes that a market equilibrium $[\mathbf{p}^*, (\mathbf{x}_i^*), \widetilde{\mathbf{y}}^*]$ in which $r_i(\mathbf{p}^*, \widetilde{\mathbf{y}}^*) = R_i(\mathbf{p}^*)$, for all i, yields an efficient allocation, no matter the pricing policies followed by the firms. Therefore, the agreement of consumers on the production plans induces efficiency whenever an equilibrium is reached. From this it follows that the usual inefficient equilibria one obtains in general equilibrium models correspond to a situations in which $\widetilde{\mathbf{y}}^* \notin \bigcap_{i=1}^m \mathcal{Y}^i(\mathbf{p}^*)$. That is, there are consumers that would like to change the production decisions in order to achieve a higher net income. This expresses the presence of a conflict between the owners' objectives and the firms' own goals.

Now consider the following axioms, that are needed in order to prove the second welfare theorem:

AXIOM 1. *For all $i = 1, 2, ..., m$,*
(i) $X_i = \mathbb{R}_+^\ell$.
(ii) $u_i : X_i \to \mathbb{R}$ *is continuous, quasi-concave, and satisfies local non-satiation.*

AXIOM 2. *For all $j = 1, 2, ..., n$, Y_j is a nonempty and closed subset of $\mathbb{R}^\ell$ such that $Y_j - \mathbb{R}_+^\ell \subset Y_j$.*

Axiom 1 establishes that every consumer is characterized by a closed convex choice set bounded from below, that we take to be $\mathbb{R}^\ell_+$ for the sake of simplicity in exposition, and a preference relation that is complete, transitive, continuous, convex and locally non-satiable. Axiom 2 refers to the firms. It postulates that production sets are closed and comprehensive (but we assume neither the convexity of choice sets nor the feasibility of inaction).

The following result is obtained:

THEOREM 2. *Under axioms 1 and 2, let* $[(\mathbf{x}_i^*), \widetilde{\mathbf{y}}^*]$ *be a Pareto optimal allocation such that* $\mathbf{x}_i^* \in int X_i$ *for all* i. *Then, there exist* $\mathbf{p}^* \in \mathbb{R}^\ell_+ - \{\mathbf{0}\}$, *and an income schedule* $(r_i)_{i=1}^m$ *such that* $[\mathbf{p}^*, (\mathbf{x}_i^*), \widetilde{\mathbf{y}}^*]$ *is a MEWAC in which firms follow the marginal pricing rule.*

Proof. First, apply the standard theorem that ensures that $[(\mathbf{x}_i^*), \widetilde{\mathbf{y}}^*]$ can be decentralized as a marginal pricing equilibrium.* This theorem proves the existence of a price vector $\mathbf{p}^* \in \mathbb{R}^\ell_+ - \{\mathbf{0}\}$ such that $[\mathbf{p}^*, (\mathbf{x}_i^*), \widetilde{\mathbf{y}}^*]$ is a marginal pricing equilibrium relative to some income distribution.

To show that $[\mathbf{p}^*, (\mathbf{x}_i^*), \widetilde{\mathbf{y}}^*]$ is a MEWAC one has to find a suitable income schedule $(r_i)_{i=1}^m$ and to check that parts (i), (ii) and (v) of Definition 5 are satisfied (parts (iii) and (iv) being satisfied by construction). Let $\beta_i(\mathbf{p})$ denote the ratio between the ith consumer's cost of acquiring $\mathbf{x}_i^*$ at prices $\mathbf{p}$, and the total worth of $\sum_{i=1}^m \mathbf{x}_i^*$ also evaluated at prices $\mathbf{p}$. That is,

$$\beta_i(\mathbf{p}) = \frac{\mathbf{p}\mathbf{x}_i^*}{\mathbf{p}\sum_{i=1}^m \mathbf{x}_i^*}$$

Now consider the following wealth function for the ith consumer, for $i = 1, 2, ..., m$:

$$r_i(\mathbf{p}, \widetilde{\mathbf{y}}) = \min \left\{ \beta_i(\mathbf{p}) \left(\sum_{j=1}^n \mathbf{p}\mathbf{y}_j + \mathbf{p}\omega \right), \quad \mathbf{p}\mathbf{x}_i^* \right\}$$

Therefore, $r_i(\mathbf{p}, \widetilde{\mathbf{y}})$ is the minimum between a share $\beta_i(\mathbf{p})$ of the aggregate wealth at $(\mathbf{p}, \widetilde{\mathbf{y}})$, and the cost of $\mathbf{x}_i^*$ at prices $\mathbf{p}$. Each r_i is clearly homogeneous of degree one in prices.

Summing over i, we get:

$$\sum_{i=1}^m r_i(\mathbf{p}, \widetilde{\mathbf{y}}) = \sum_{i=1}^m \min \left\{ \mathbf{p}\mathbf{x}_i^* \frac{\sum_{j=1}^n \mathbf{p}\mathbf{y}_j + \mathbf{p}\omega}{\mathbf{p}\sum_{i=1}^m \mathbf{x}_i^*}, \quad \mathbf{p}\mathbf{x}_i^* \right\}$$

Now observe that if $\mathbf{p}(\sum_{j=1}^n \mathbf{y}_j + \omega) \geqslant \mathbf{p}\sum_{i=1}^m \mathbf{x}_i^*$, all the terms of the sum take on the value $\mathbf{p}\mathbf{x}_i^*$ so that $\sum_{i=1}^m r_i(\mathbf{p}, \widetilde{\mathbf{y}}) = \mathbf{p}\sum_{i=1}^m \mathbf{x}_i^* \leqslant \mathbf{p}(\sum_{j=1}^n \mathbf{y}_j + \omega)$. If, alternatively, $\mathbf{p}(\sum_{j=1}^n \mathbf{y}_j + \omega) < \mathbf{p}\sum_{i=1}^m \mathbf{x}_i^*$, then $\sum_{i=1}^m r_i(\mathbf{p}, \widetilde{\mathbf{y}}) = \mathbf{p}(\sum_{j=1}^n \mathbf{y}_j + \omega)$.

* Detailed proofs of this theorem appear in Quinzii (1992, ch. 2) and Villar (2000a, ch. 6).

Therefore, $\sum_{i=1}^{m} r_i(\mathbf{p}, \widetilde{\mathbf{y}}) \leqslant \sum_{j=1}^{n} \mathbf{p}\mathbf{y}_j + \mathbf{p}\omega$, that is, $(r_i)_{i=1}^{m}$ is an income schedule, according to definition 2. Moreover, when evaluated at $(\mathbf{p}^*, \widetilde{\mathbf{y}}^*)$, we get:

$$r_i(\mathbf{p}^*, \widetilde{\mathbf{y}}^*) = \min \left\{ \mathbf{p}^*\mathbf{x}_i^* \frac{\sum_{j=1}^{n} \mathbf{p}^*\mathbf{y}_j^* + \mathbf{p}^*\omega}{\mathbf{p}^* \sum_{i=1}^{m} \mathbf{x}_i^*}, \quad \mathbf{p}\mathbf{x}_i^* \right\} = \mathbf{p}^*\mathbf{x}_i^*$$

Let $\pi : \mathbb{R}_+^{\ell} \to \mathbb{R}$ be a mapping given by $\pi(\mathbf{p}) = \sup_{\widetilde{\mathbf{y}} \in \mathbb{F}^{\mathcal{A}}} \sum_{j=1}^{m} \mathbf{p}\mathbf{y}_j$, with $\pi(\mathbf{p}) = +\infty$ if no maximum exists at prices $\mathbf{p}$. The function R_i associated with the income mapping r_i is given by:

$$R_i(\mathbf{p}) = \min \left\{ \beta_i(\mathbf{p}) \left[\pi(\mathbf{p}) + \mathbf{p}\omega \right], \quad \mathbf{p}\mathbf{x}_i^* \right\}$$

Clearly, if $\beta_i(\mathbf{p})[\pi(\mathbf{p}) + \mathbf{p}\omega] > \mathbf{p}\mathbf{x}_i^*$ it follows that $r_i(\mathbf{p}, \widetilde{\mathbf{y}}) = R_i(\mathbf{p}) = \mathbf{p}\mathbf{x}_i^*$. Suppose now that $\beta_i(\mathbf{p})[\pi(\mathbf{p}) + \mathbf{p}\omega] \leqslant \mathbf{p}\mathbf{x}_i^*$. We can rewrite this expression as:

$$\pi(\mathbf{p}) + \mathbf{p}\omega \leqslant \frac{1}{\beta_i(\mathbf{p})} \mathbf{p}\mathbf{x}_i^* = \mathbf{p}\left(\sum_{j=1}^{n} \mathbf{y}_j^* + \omega\right)$$

which is possible only if $\pi(\mathbf{p}) + \mathbf{p}\omega = \mathbf{p}(\sum_{j=1}^{n} \mathbf{y}_j^* + \omega)$, by the very definition of $\pi(\mathbf{p})$. Therefore, when evaluated at $(\mathbf{p}^*, \widetilde{\mathbf{y}}^*)$ we find that $r_i(\mathbf{p}^*, \widetilde{\mathbf{y}}^*) = R_i(\mathbf{p}^*) = \mathbf{p}^*\mathbf{x}_i^*$. Therefore, $\widetilde{\mathbf{y}}^* \in \bigcap_{i=1}^{m} \mathcal{Y}^i(\mathbf{p}^*)$ and parts (ii) and (v) of definition 5 are satisfied.

Finally, take a consumer i and a consumption plan $\mathbf{x}_i \in X_i$ such that $u_i(\mathbf{x}_i) > u_i(\mathbf{x}_i^*)$. It is immediate to see that this consumption plan is not affordable because, by definition, $\mathbf{p}^*\mathbf{x}_i > R_i(\mathbf{p}^*) = \mathbf{p}^*\mathbf{x}_i^*$, which is the minimum expenditure that is required to attain a utility greater than or equal to $u_i(\mathbf{x}_i^*)$ at prices $\mathbf{p}^*$. From this and the interiority assumption it is routine to show that part (i) of the definition is also satisfied, so that the proof is completed. □

Theorem 2 establishes that, under fairly general assumptions, any Pareto efficient allocation can be decentralized as a MEWAC in which firms follow the marginal pricing rule. Note that when u_i is differentiable for some i, on a neighborhood of $\mathbf{x}_i^* \in int X_i$, the (normalized) vector of marginal rates of substitution is unique, so that the (normalized) price vector supporting that allocation is unique as well.⋆ This amounts to saying that a market equilibrium is efficient only if it corresponds to a MEWAC in which firms follow the marginal pricing rule.

Some remarks are in order:

(i) Taking $\mathbf{x}_i^* \in int X_i$ for all i is too strong an assumption (that is used here for the sake of simplicity in exposition). The only thing which is required in order to

⋆ This implies that $\frac{\partial u_i/\partial x_{ik}}{\partial u_i/\partial x_{ih}} = \frac{p_k^*}{p_h^*}$, for all $k, h = 1, 2, ..., \ell$, is a necessary condition for the efficiency of market equilibria. This condition is naturally satisfied in our model because $\frac{\partial r_i(.)}{\partial x_{ik}} = 0$, according to definition 2 (i.e. in an interior allocation the income function is independent of the consumption level of x_{ik}).

derive utility maximization from expenditure minimization is that $\mathbf{p}^*\mathbf{x}_i^* > 0$ for all i.

(ii) When there are commodities that do not enter the preferences of consumers, the efficient equilibrium price vector must be a marginal pricing vector in the sub-space of commodities that are effectively consumed, and we find some degrees of freedom in the complementary subspace.

(iii) When firms experience quantity constraints, the restriction imposed on firms by marginal prices is less tight because the cone of normals at $\mathbf{y}_j$ in the truncated production set is larger than the usual normal cone to Y_j at the boundary point $\mathbf{y}_j$.

(iv) This result can be extended further by using Mordukhovich normal cones rather than Clarke normal cones [see the exposition in Khan (1999) and the original contribution in Mordukhovich (1976)].

3.2. MEWAC AND MARGINAL PRICING

Let us illustrate the relationship between marginal pricing equilibrium and ME-WAC by means of a simple example.

Consider a private ownership economy with two goods, labour and corn, two identical consumers, and two firms. Each consumer is endowed with a unit of labour, that is supplied inelastically and owns $1/2$ of each firm. Preferences are strictly monotone in corn. The ith consumer's income function is given by $r_i(\mathbf{p}, \widetilde{\mathbf{y}}) = p_1 + \frac{1}{2}(\mathbf{p}\mathbf{y}_1 + \mathbf{p}\mathbf{y}_2)$.

Firm 1 exhibits constant returns to scale, with efficient production plans of the form $\mathbf{y}_1 = \lambda(-1,\ 1)$. The production set of firm 2 has a piece-wise linear frontier in the following sense. It coincides with that of firm 1 for those production plans involving less labour than 1,5 units, and becomes more productive for higher levels of output, with a slope of -2. The next figure illustrates these production sets.

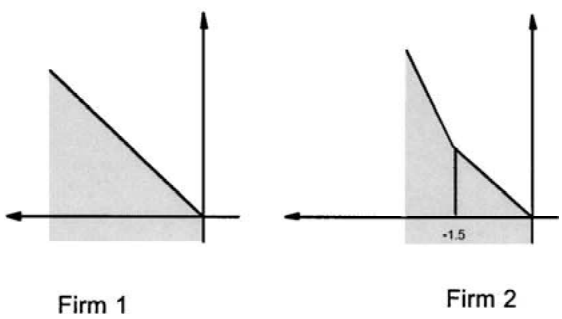

Figure 1.

Now observe that the price vector $\mathbf{p}' = (1,\ 1)$, together with the allocation $\mathbf{x}_i' = (-1, 1)$ for $i = 1, 2$, and $\mathbf{y}_j = (-1, 1)$ for $j = 1, 2$, is a market equilibrium in which firms follow the marginal pricing rule (which here coincides with average cost pricing). This is *not* a MEWAC because at these prices the consumers would obtain a higher income by closing down firm 1 and applying the two units of labour to firm 2. That option would yield an additional income of 0.25 units per head. The price vector $\mathbf{p}^* = (1, 0.8)$, and the allocation $\mathbf{y}_1^* = \mathbf{0}$, $\mathbf{y}_2^* = (-2, 2.5)$, $\mathbf{x}_i^* = (-1, 1.25)$ for $i = 1, 2$ is a MEWAC in which firms follow the average cost pricing rule.*

The inefficiency of marginal pricing in this example can be related to the presence of a wrong number of active firms. Extending the scope of consumers' actions, as suggested by the notion of MEWAC, permits to eliminate this inefficiency. Yet, let us recall here that Guesnerie (1975) and Brown and Heal (1979) present examples of economies with a single firm in which no marginal pricing equilibrium is efficient (see Villar (2000a, ch. 6) for a discussion). This happens for some income schedules which are inherently incompatible with efficiency. What this implies for our model is that in these economies there is *no* MEWAC in which firms follow the marginal pricing rule, relative to that income schedule.

4. Final Comments

We have shown in the former sections that giving a more active role to the consumers in production decisions permits one to ensure the efficiency of equilibrium outcomes (Theorem 1). And also that each efficient allocation corresponds to a MEWAC in which firms follow the marginal pricing rule (Theorem 2).

It follows from those results that the efficiency of market equilibria calls for two restrictions to be satisfied:

(i) The equilibrium allocation must be supportable as a marginal pricing equilibrium. This restriction introduces local properties on the relationship between agents' choices and equilibrium price systems.

(ii) The income schedule must be rich enough to induce global optimization. This is a global condition on the income generated by the economy.

From this it follows that he existence of a MEWAC depends both on the flexibility of the income schedule and the compatibility between the consumers' production decisions and the firms' pricing rules. Since a MEWAC is a market equilibrium $[\mathbf{p}^*, (\mathbf{x}_i^*), \widetilde{\mathbf{y}}^*]$ in which $\widetilde{\mathbf{y}}^* \in \bigcap_{i=1}^m \mathcal{Y}^i(\mathbf{p}^*)$, the set of economies for which a MEWAC exists is smaller than those for which one can ensure the existence of market equilibria. In a standard private ownership market economy those two notions coincide, because marginal pricing implies profit maximization when production sets are convex and because the competitive income mappings are precisely the functions R_i (as firms maximize profits at given prices and there are neither taxes nor subsidies). But we cannot count on this in general. Only particular income schedules can ensure that $r_i(\mathbf{p}^*, \widetilde{\mathbf{y}}^*) = R_i(\mathbf{p}^*)$ for all i, when firms are not price-taking profit-maximizing entities. Therefore, the usual conditions under which the

* It is easy to see that there is also a marginal pricing equilibrium associated with this allocation and the same income rule, at prices $\widehat{\mathbf{p}} = (1, 0.2)$.

existence of market equilibrium is postulated, typically valid for any arbitrary given income schedule, may well be incompatible with the efficiency requirements.

There are some particular models of imperfectly competitive economies in which the existence of a MEWAC can be ensured. This is the case in the following examples:

(1) Models with a single firm that follows marginal pricing and “suitable” consumers (Brown and Heal, 1979; Vohra, 1988; Quinzii, 1991).

(2) Models with particular types of marginal pricing, such as two-part tariffs (Brown et al., 1992; Moriguchi, 1996), other forms of non-linear marginal pricing (Vega-Redondo, 1987), or “personalized commodities” (Edlin et al., 1998).

(3) Models in which firms maximize profits subject to an input restriction (Scarf, 1986; Villar, 2000a, chs. 8, 9, 2000b).

Our analysis also suggests that when firms’ decisions are independent on the consumers’ interests, efficiency requires private wealth to be supplemented by a suitable system of taxes and transfers. Clearly the presence of a tax-subsidy rule does not ensure efficiency (the inefficiency of marginal pricing equilibria is well known). But without such a system there is little hope of achieving efficient outcomes through a market mechanism. To put it in a more provocative way: those market mechanisms in which consumers do not control the firms are generally inefficient because, except in the extremely rare case of perfect competition, some public intervention is necessary (though not sufficient) for the achievement of optimal allocations. Note that the key purpose of this tax-subsidy scheme would be to induce the right allocation of resources, rather than performing a redistribution policy.

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